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Generalizations of the Carlton-Kimball Distribution for a Target's Future Location

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Abstract—This paper addresses the problem of searching for a target in the plane given only an initial position estimate and bounds on its speed and direction. Exact results are derived for determining the position density of the target's future location assuming constant linear motion. In particular, we derive a generalization of the Carlton-Kimball distribution.

Keywords—Search theory, Applications of hypergeometric functions.

1. INTRODUCTION

During the 1940's Koopman, in a series of reports which were later published as a book [1], laid the scientific foundations for the mathematical field of search theory. One of the simplest problems in search theory is that of determining (probabilistically) the future position of a target given a probability distribution for its initial position, and a distribution for its fixed velocity. Under certain restrictive conditions Koopman determined asymptotically the target's future location.

In [2], Moskowitz and Simmen generalized Koopman's work. Given an initial bivariate normal distribution of a target's location and general distributions (of independent magnitude and direction) for its constant velocity, they described the target's future location asymptotically. Some of these results were used to discuss various search strategies.

In the present paper, we shall extend this work somewhat further to represent, under certain conditions, the position density of the target's future location exactly in closed form by employing certain easily computable special functions. In particular, we derive a generalization of the Carlton-Kimball distribution.

The results presented herein have application to the problem of planning a search for a target whose position has been observed with probable error exactly once. The situation of using multiple-observations to forecast future target position is handled by the well-known methods of Kalman filtering (see, e.g., [3,4]). In this paper, we restrict our discussion to single target observations.

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2. DETERMINISTIC MOTION

In general, a target's motion in two dimensions is governed by its time dependent velocity field $\vec{v}(x, y, t)$. If the target's initial position is given probabilistically by the random vector $\mathbf{X}_0$, we let $\mathbf{X}_t$ represent the random vector describing its future position at time t. The positional density function corresponding to $\mathbf{X}_t$ is denoted by $p_t(x, y)$ or also by $p(x, y, t)$. Consider a region R_0 in the plane. Let R_t be the region that evolves from R_0 via the (assumed well-defined and invertible [1, p. 119]) flow of $\vec{v}(x, y, t)$. The probability of containment is attached to a region and changes with the region. By this we mean that $P(\mathbf{X}_t \in R_t) = P(\mathbf{X}_0 \in R_0)$. In integral form, we can state this as

$$\frac{d}{dt} \int_{R(t)} p_t(x, y)\, dA = 0. \tag{1}$$

By applying the transport theorem [5, p. 469] in the plane to equation (1) we arrive at

$$\int_{R(t)} \left(\frac{\partial p}{\partial t} + \nabla \cdot (p\, \vec{v}) \right) dA = 0,$$

where the vector operator $\nabla = \frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath}$. Since the above equation holds for all sufficiently regular regions $R(t)$, we arrive at the "conservation of probability" equation (cf. the continuity equation in [1, p. 121])

$$\frac{\partial p}{\partial t} + \nabla \cdot (p\, \vec{v}) = 0. \tag{2}$$

The above analysis applies when $\vec{v}$ is deterministic. If the initial probability density is known, then $p_t(x, y)$ can always be computed, perhaps numerically if not analytically, via equation (2). This is so because the initial probability density gives us the boundary condition $p(x, y, 0) = p_0(x, y)$.

If the velocity, $\vec{v} = v_x\vec{\imath} + v_y\vec{\jmath}$, is constant, then equation (2) reduces to

$$\frac{\partial p}{\partial t} + v_x \frac{\partial p}{\partial x} + v_y \frac{\partial p}{\partial y} = 0. \tag{3}$$

By making the transformation of variables $(x, y, t) \to (x - v_x t, x + v_x t, y - v_y t)$, or by using the method of characteristics, we arrive at the solution of equation (3):

$$p(x, y, t) = p_0(x - tv_x, y - tv_y). \tag{4}$$

In terms of random vectors, this is just $\mathbf{X}_t = \mathbf{X}_0 + t\vec{v}$. This means that for a constant velocity field the initial density is just translated by a vector equal to the velocity times the elapsed time t.

Now let ν be the target's speed and ϕ be the direction of motion off the x-axis. Hence, $v_x = \nu\cos\phi$, $v_y = \nu\sin\phi$, and equation (4) yields

$$p_t(x, y) = p_0(x - t\nu\cos\phi, y - t\nu\sin\phi). \tag{5}$$

Thus, we shall be able to deal with situations where we have information about a target's speed and heading.

3. FIXED NONDETERMINISTIC MOTION

Now the velocity field is fixed, but unknown. In other words, the target's motion is constant but determined by some random vector $\mathbf{V}$. Hence, the location of the target at future time t is given by $\mathbf{X}_t = \mathbf{X}_0 + t\mathbf{V}$. In fact $\mathbf{V}$ can be viewed as a function of the jointly distributed random variables $\boldsymbol{\nu}$ and $\boldsymbol{\phi}$ which describe the speed and direction of the target; i.e., $\mathbf{V} = \boldsymbol{\nu}\cos\boldsymbol{\phi}\vec{\imath} + \boldsymbol{\nu}\sin\boldsymbol{\phi}\vec{\jmath}$.

By conditioning on $\boldsymbol{\nu}$ and $\boldsymbol{\phi}$, and using equation (5), we obtain

$$p_t(x, y) = \int_0^\infty \int_0^{2\pi} p_0(x - t\nu\cos\phi,\; y - t\nu\sin\phi)\; p(\phi, \nu)\; d\phi\; d\nu,$$

where $p(\phi, \nu)$ is the joint density of ϕ and ν. In particular, assuming that ν and ϕ are independent with the probability density functions $p_1(\nu)$ and $p_2(\phi)$, respectively, we obtain

$$p_t(x, y) = \int_{\nu_1}^{\nu_2} \int_{\phi_1}^{\phi_2} p_0(x - t\nu\cos\phi,\ y - t\nu\sin\phi)\ p_1(\nu)\ p_2(\phi)\ d\phi\ d\nu, \tag{6}$$

where $[\nu_1,\ \nu_2]$ is the range of ν, $[\phi_1,\ \phi_2]$ is the range of ϕ, and $p_1(\nu)$ and $p_2(\phi)$ are assumed to vanish outside their respective domains of definition. The above equation is quite general. In certain cases exact analytic results can be derived from it, but the situations of most interest heretofore have been solved numerically or approximated asymptotically (see [1,2]).

A useful equivalent form of equation (6) may be given by switching to polar coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{7}$$

The density function $p_t(x, y)$ in the new polar coordinate system (r, θ) will be denoted by $\hat{p}_t(r, \theta)$. Thus, it is easy to see from equation (6) that

$$\hat{p}_t(r, \theta) = r \int_{\nu_1}^{\nu_2} \int_{\phi_1}^{\phi_2} p_0(r\cos\theta - t\nu\cos\phi, r\sin\theta - t\nu\sin\phi)\ p_1(\nu)\ p_2(\phi)\ d\phi\ d\nu, \tag{8}$$

since the Jacobian of the transformation (7) is equal to r.

4. THE CASE OF INITIAL POSITION GIVEN BY A CIRCULAR NORMAL DISTRIBUTION

The situation that we are interested in here has three assumptions: the initial position of the target is given by a normal distribution that is circular about the origin with variance σ^2, the speed is uniform between ν_1 and ν_2, and the course distribution is uniform in all directions. Thus,

$$\begin{aligned} p_0(x, y) &= \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right), & |x|, |y| < \infty, \\ p_1(\nu) &= \frac{1}{\nu_2 - \nu_1}, & \nu_1 \le \nu \le \nu_2, \\ p_2(\phi) &= \frac{1}{2\pi}, & 0 \le \phi < 2\pi. \end{aligned}$$

Now by using these densities together with equation (8) we obtain, after simplification, the result

$$\hat{p}_t(r, \theta) = \frac{r\exp(-r^2/2\sigma^2)}{2\pi\sigma^2(\nu_2 - \nu_1)} \int_{\nu_1}^{\nu_2} \left[\frac{1}{2\pi} \int_0^{2\pi} \exp\left(\frac{r\nu t}{\sigma^2}\cos(\phi - \theta)\right) d\phi\right] \exp\left(\frac{-\nu^2 t^2}{2\sigma^2}\right) d\nu. \tag{9}$$

It is well known that the inner integral for $\theta = 0$ is the zeroth order modified Bessel function (see, e.g., [6, Section 8.431, equation (3)]) and it is not difficult to show more generally that

$$I_0(z) = \frac{1}{2\pi} \int_0^{2\pi} \exp(z\cos(\phi - \theta))\ d\phi.$$

Thus equation (9) simplifies to

$$\hat{p}_t(r, \theta) = \frac{r\exp(-r^2/2\sigma^2)}{2\pi\sigma^2(\nu_2 - \nu_1)} \int_{\nu_1}^{\nu_2} \exp\left(\frac{-\nu^2 t^2}{2\sigma^2}\right) I_0\left(\frac{r\nu t}{\sigma^2}\right) d\nu; \tag{10}$$

and since $\nu_1 < \nu_2$, on making a change of variables, we see that

$$\hat{p}_t(r, \theta) = \frac{r\exp\left(-r^2/2\sigma^2\right)}{2\pi rt(\nu_2 - \nu_1)} \left[I_{e_0^2}\left(\frac{-\sigma^2}{2r^2}, \frac{rtv_2}{\sigma^2}\right) - I_{e_0^2}\left(\frac{-\sigma^2}{2r^2}, \frac{rtv_1}{\sigma^2}\right)\right], \tag{11}$$

where

$$I_{e_0^2}(a,z) \equiv \int_0^z \exp\left(a\tau^2\right) I_0(\tau)\, d\tau. \tag{12}$$

Thus, equations (10) and (11) give the future position density when the initial target location is described by a circular normal distribution centered about the origin, the target heading is uniformly likely in all directions, and the target speed is uniformly likely between ν_1 and ν_2. If the speed is known exactly as ν_0, then by letting $\nu_1, \nu_2 \to \nu_0$, equation (10) yields via the mean value theorem for integration

$$\hat{p}_t(r,\theta) = \frac{r}{2\pi\sigma^2} \exp\left[\frac{-\left(r^2+\nu_0^2t^2\right)}{2\sigma^2}\right] I_0\left(\frac{r\nu_0 t}{\sigma^2}\right). \tag{13}$$

Koopman, who derives equation (13) in a different way [1, p. 125], attributes this density to G. E. Kimball (circa 1943) and mentions that it is also known as Carlton's distribution [1, p. 127]. We shall derive a much more general result in Section 6 which includes the Carlton-Kimball distribution as a special case (see equations (21) and (22)). The results given by equations (8)–(11) and (13) incidentally clarify a missing factor of r in the formulas for $\hat{p}_t(r,\theta)$ previously given in [1,2,7].

Although the future position density in equation (10) is given exactly in terms of a single definite integral whose positive integrand is rather simple, this integral can (unfortunately) not be evaluated in terms of elementary functions or even known higher transcendental functions of a single variable.

Thus, in order to evaluate the position density, we must either use a suitable computational routine for numerical quadrature or resort to an evaluation utilizing other known appropriate special functions. Fortunately, such a class of functions, which are known as generalized hypergeometric functions in several variables, is by now well developed and understood from both a theoretical and computational point of view.

In addition, as is well known, representation by special functions is not only superior for further theoretical developments (as we shall see in Section 6), but is generally less costly in terms of computational resources *vis-à-vis* numerical quadrature. With this in mind we shall continue our study of the position density in the settings of Kampé de Fériet functions (in Section 5) and Srivastava's $F^{(3)}$-function (in Section 6) which are, respectively, generalized hypergeometric functions in two and three variables.

5. EXACT RESULTS FOR THE POSITION DENSITY $\hat{p}_t(r,\theta)$

The function $I_{e_0^2}(a,z)$ defined by equation (12) is an example of an incomplete Weber integral. Such integrals have already been extensively treated in [7]. Thus from [7, equation (3.7)] we have

$$I_{e_0^2}(a,z) = z\exp\left(az^2\right) F^{0:1;1}_{1:1;0}\left[\begin{matrix} \text{---}: & 1/2; & 1; \\ 3/2: & 1; & \text{---}; \end{matrix}\ \frac{z^2}{4}, -az^2\right], \tag{14}$$

where $F^{0:1;1}_{1:1;0}[x,y]$ is a special case of the so-called Kampé de Fériet functions defined by (see, e.g., [8])

$$F^{p:q\,;k}_{l:m;n}\left[\begin{matrix} (a_p): & (b_q); & (c_k); \\ (\alpha_l): & (\beta_m); & (\gamma_n); \end{matrix}\ x,\ y\right] \equiv \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{r+s} \prod_{j=1}^{q}(b_j)_r \prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s} \prod_{j=1}^{m}(\beta_j)_r \prod_{j=1}^{n}(\gamma_j)_s} \frac{x^r}{r!} \frac{y^s}{s!},$$

where the Pochhammer symbol, or factorial function, $(a)_n$ is defined by $(a)_n \equiv \Gamma(a+n)/\Gamma(a)$ for nonnegative integers n and (a_p) represents a p-tuple of complex numbers. The exact region of convergence for Kampé de Fériet functions, and for its generalization $F^{(3)}$ (which we shall use in Section 6), is determined, respectively, by using Horn's theorems for double and triple series [9,10].

Thus from equations (11) and (14) we deduce an exact result for the position density:

$$\hat{p}_t(r,\theta) = \frac{r\exp(-r^2/2\sigma^2)}{2\pi\sigma^2(\nu_2-\nu_1)}\left\{\nu_2\exp\left(\frac{-\nu_2^2t^2}{2\sigma^2}\right)F^{0:1;1}_{1:1;0}\left[\begin{matrix} \text{---}: & 1/2; & 1; \\ 3/2: & 1; & \text{---}; \end{matrix}\ \frac{r^2\nu_2^2t^2}{4\sigma^4}, \frac{\nu_2^2t^2}{2\sigma^2}\right]\right.$$
$$\left. - \nu_1\exp\left(\frac{-\nu_1^2t^2}{2\sigma^2}\right)F^{0:1;1}_{1:1;0}\left[\begin{matrix} \text{---}: & 1/2; & 1; \\ 3/2: & 1; & \text{---}; \end{matrix}\ \frac{r^2\nu_1^2t^2}{4\sigma^4}, \frac{\nu_1^2t^2}{2\sigma^2}\right]\right\}.$$

Since also from [7, equation (3.4)]

$$I_{e_0^2}(a,z) = zF^{1:0;0}_{1:1;0}\left[\begin{matrix} 1/2: & \text{---}; & \text{---}; \\ 3/2: & 1; & \text{---}; \end{matrix}\ \frac{z^2}{4}, az^2\right]$$

we have in addition from equation (11)

$$\hat{p}_t(r,\theta) = \frac{r\exp(-r^2/2\sigma^2)}{2\pi\sigma^2(\nu_2-\nu_1)}\left\{\nu_2F^{1:0;0}_{1:1;0}\left[\begin{matrix} 1/2: & \text{---}; & \text{---}; \\ 3/2: & 1; & \text{---}; \end{matrix}\ \frac{r^2\nu_2^2t^2}{4\sigma^4}, \frac{-\nu_2^2t^2}{2\sigma^2}\right]\right.$$
$$\left. - \nu_1F^{1:0;0}_{1:1;0}\left[\begin{matrix} 1/2: & \text{---}; & \text{---}; \\ 3/2: & 1; & \text{---}; \end{matrix}\ \frac{r^2\nu_1^2t^2}{4\sigma^4}, \frac{-\nu_1^2t^2}{2\sigma^2}\right]\right\}.$$

Yet another representation for the position density may be obtained by using [7, equation (2.4)] together with equation (11); thus since $I_{-1}(z) = I_1(z)$, we obtain

$$\hat{p}_t(r,\theta) = \frac{r\exp(-r^2/2\sigma^2)}{2\pi\sigma^2}\frac{\nu_2}{\nu_2-\nu_1}$$
$$\cdot\left\{I_0\left(\frac{rt\nu_2}{\sigma^2}\right)F^{0:2;1}_{2:0;0}\left[\begin{matrix} \text{------}: & 1/2,1/2; & 1; \\ 1/2,3/2: & \text{------}; & \text{---}; \end{matrix}\ \frac{-\nu_2^2t^2}{2\sigma^2}, \frac{r^2\nu_2^2t^2}{4\sigma^4}\right]\right.$$
$$\left. - \frac{rt\nu_2}{\sigma^2}I_1\left(\frac{rt\nu_2}{\sigma^2}\right)F^{0:2;1}_{2:0;0}\left[\begin{matrix} \text{------}: & 1/2,1/2; & 1; \\ 3/2,3/2: & \text{------}; & \text{---}; \end{matrix}\ \frac{-\nu_2^2t^2}{2\sigma^2}, \frac{r^2\nu_2^2t^2}{4\sigma^4}\right]\right\}$$
$$- \frac{r\exp(-r^2/2\sigma^2)}{2\pi\sigma^2}\frac{\nu_1}{\nu_2-\nu_1}$$
$$\cdot\left\{I_0\left(\frac{rt\nu_1}{\sigma^2}\right)F^{0:2;1}_{2:0;0}\left[\begin{matrix} \text{------}: & 1/2,1/2; & 1; \\ 1/2,3/2: & \text{------}; & \text{---}; \end{matrix}\ \frac{-\nu_1^2t^2}{2\sigma^2}, \frac{r^2\nu_1^2t^2}{4\sigma^4}\right]\right.$$
$$\left. - \frac{rt\nu_1}{\sigma^2}I_1\left(\frac{rt\nu_1}{\sigma^2}\right)F^{0:2;1}_{2:0;0}\left[\begin{matrix} \text{------}: & 1/2,1/2; & 1; \\ 3/2,3/2: & \text{------}; & \text{---}; \end{matrix}\ \frac{-\nu_1^2t^2}{2\sigma^2}, \frac{r^2\nu_1^2t^2}{4\sigma^4}\right]\right\}.$$

We remark that the Kampé de Fériet functions employed in the latter three representations for $\hat{p}_t(r,\theta)$ converge everywhere in their two independent variables.

6. THE CASE OF INITIAL POSITION GIVEN BY A GENERAL BIVARIATE NORMAL DISTRIBUTION

In Section 4, we derived the density function of a target's location at future time $t > 0$ assuming a circular normal distribution for its initial location. We shall now generalize this result by assuming a general bivariate normal distribution for the initial position, *viz.*

$$p_0(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}$$
$$\cdot\exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]\right\}, \quad |x|,|y| < \infty$$

where μ_x, μ_y and σ_x^2, σ_y^2 are, respectively, the means and variances of x, y, and ρ is the correlation coefficient of x and y.

It can be shown by a suitable transformation of the coordinates (x, y), on taking into account the Jacobian of this transformation, that $p_0(x, y)$ can be put in the form

$$p_0(x,y) = \frac{1}{2\pi\sigma_x\sigma_y} \exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right)\right], \qquad |x|, |y| < \infty$$

where it should be noted that since we are in a new reference system, the variances σ_x^2 and σ_y^2 have been adjusted accordingly.

Now by assuming, as in the derivation of equation (9), a uniform target speed between ν_1 and ν_2 and a uniform course distribution, from equation (6) we arrive at

$$\begin{aligned} p_t(x,y) = &\frac{1}{(2\pi)^2\sigma_x\sigma_y(\nu_2-\nu_1)} \\ &\cdot \int_{\nu_1}^{\nu_2}\int_0^{2\pi} \exp\left\{-\frac{1}{2}\left[\left(\frac{x-t\nu\cos\phi}{\sigma_x}\right)^2 + \left(\frac{y-t\nu\sin\phi}{\sigma_y}\right)^2\right]\right\} d\phi\, d\nu. \end{aligned} \tag{15}$$

If we define

$$J(\nu) \equiv \int_0^{2\pi} \exp\left(a\nu^2\sin^2\phi\right)\exp(\alpha\nu\cos\phi)\exp(\beta\nu\sin\phi)\, d\phi, \tag{16}$$

where

$$a \equiv \frac{t^2}{2}\left(\frac{1}{\sigma_x^2} - \frac{1}{\sigma_y^2}\right)$$

and

$$\alpha \equiv \frac{tx}{\sigma_x^2}, \qquad \beta \equiv \frac{ty}{\sigma_y^2},$$

we may rewrite equation (15), after some computation, as

$$p_t(x,y) = \frac{\exp\left[-(1/2)\left(x^2/\sigma_x^2 + y^2/\sigma_y^2\right)\right]}{(2\pi)^2\sigma_x\sigma_y(\nu_2-\nu_1)} \int_{\nu_1}^{\nu_2} \exp\left(\frac{-t^2\nu^2}{2\sigma_x^2}\right) J(\nu)\, d\nu. \tag{17}$$

Next, we evaluate $J(\nu)$ defined by equation (16). Writing each exponential in the integrand as a Maclaurin series, and interchanging the resulting three summations and integral, it is easily seen that

$$J(\nu) = \sum_{m,n,p=0}^{\infty} \frac{(a\nu^2)^m}{m!}\frac{(\alpha\nu)^n}{n!}\frac{(\beta\nu)^p}{p!}\int_0^{2\pi} \sin^{2m+p}\phi\cos^n\phi\, d\phi. \tag{18}$$

Noting [6, Section 3.621, equation (5)]

$$\int_0^{\frac{\pi}{2}} \sin^{\mu-1}\phi\cos^{\nu-1}\phi\, d\phi = \frac{1}{2}\frac{\Gamma(\mu/2)\Gamma(\nu/2)}{\Gamma((\mu+\nu)/2)}, \qquad \mathrm{Re}\,\mu, \mathrm{Re}\,\nu > 0$$

we observe, after some computation, that the integral in equation (18) vanishes unless n and p are even integers, in which case we obtain

$$J(\nu) = 2\pi \sum_{m,n,p=0}^{\infty} \frac{(a\nu^2)^m}{m!}\frac{(\alpha^2\nu^2/4)^n}{n!}\frac{(\beta^2\nu^2/4)^p}{p!}\frac{(1/2)_{m+p}}{(1/2)_p(1)_{m+n+p}}.$$

Thus, by using Srivastava's $F^{(3)}$-function [8, p. 69, equation (39)] we deduce

$$J(\nu) = 2\pi F^{(3)}\left[\begin{matrix} - & :: & -; & -; & 1/2: & -; & -; & -; \\ 1 & :: & -; & -; & -: & -; & -; & 1/2; \end{matrix}\; a\nu^2, \frac{\alpha^2\nu^2}{4}, \frac{\beta^2\nu^2}{4}\right],$$

where $F^{(3)}$ converges everywhere in its three independent variables. Hence from equation (17) we arrive at

$$\begin{aligned} p_t(x,y) &= \frac{\exp\left[-(1/2)\left(x^2/\sigma_x^2+y^2/\sigma_y^2\right)\right]}{2\pi\sigma_x\sigma_y(\nu_2-\nu_1)} \\ &\quad \cdot \int_{\nu_1}^{\nu_2} \exp\left(\frac{-t^2\nu^2}{2\sigma_x^2}\right) F^{(3)}\left[\left(\frac{1}{\sigma_x^2}-\frac{1}{\sigma_y^2}\right)\frac{t^2\nu^2}{2}, \frac{t^2\nu^2x^2}{4\sigma_x^4}, \frac{t^2\nu^2y^2}{4\sigma_y^4}\right] d\nu, \end{aligned} \tag{19}$$

where for conciseness we have omitted the parameters in the $F^{(3)}$-function. We may similarly express equation (17) as

$$\begin{aligned} p_t(x,y) &= \frac{\exp\left[-(1/2)\left(x^2/\sigma_x^2+y^2/\sigma_y^2\right)\right]}{2\pi\sigma_x\sigma_y(\nu_2-\nu_1)} \\ &\quad \cdot \int_{\nu_1}^{\nu_2} \exp\left(\frac{-t^2\nu^2}{2\sigma_y^2}\right) F^{(3)}\left[\left(\frac{1}{\sigma_y^2}-\frac{1}{\sigma_x^2}\right)\frac{t^2\nu^2}{2}, \frac{t^2\nu^2y^2}{4\sigma_y^4}, \frac{t^2\nu^2x^2}{4\sigma_x^4}\right] d\nu. \end{aligned} \tag{20}$$

Now letting $\nu_1, \nu_2 \to \nu_0$ we obtain generalizations of equation (13) from equations (19) and (20), respectively, by appealing to the mean value theorem for integration. Thus, we obtain the elegant results

$$\begin{aligned} p_t(x,y) &= \frac{1}{2\pi\sigma_x\sigma_y} \exp\left(\frac{-t^2\nu_0^2}{2\sigma_x^2}\right) \exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}\right)\right] \\ &\quad \cdot F^{(3)}\left[\left(\frac{1}{\sigma_x^2}-\frac{1}{\sigma_y^2}\right)\frac{t^2\nu_0^2}{2}, \frac{t^2\nu_0^2x^2}{4\sigma_x^4}, \frac{t^2\nu_0^2y^2}{4\sigma_y^4}\right] \end{aligned} \tag{21}$$

and

$$\begin{aligned} p_t(x,y) &= \frac{1}{2\pi\sigma_x\sigma_y} \exp\left(\frac{-t^2\nu_0^2}{2\sigma_y^2}\right) \exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}\right)\right] \\ &\quad \cdot F^{(3)}\left[\left(\frac{1}{\sigma_y^2}-\frac{1}{\sigma_x^2}\right)\frac{t^2\nu_0^2}{2}, \frac{t^2\nu_0^2y^2}{4\sigma_y^4}, \frac{t^2\nu_0^2x^2}{4\sigma_x^4}\right]. \end{aligned} \tag{22}$$

That equations (21) and (22) reduce to equation (13) may be seen by letting $\sigma_x = \sigma_y = \sigma$ and using the reduction formula [9, p. 28, equation (30)]

$$F^{p:0;0}_{q:0;0}\left[\begin{matrix}(a_p): & -; & -; \\ (b_q): & -; & -;\end{matrix}\; u,\ v\right] = {}_pF_q[(a_p);\ (b_q);\ u+v].$$

Thus, since ${}_0F_1[-;\ 1;\ z^2/4] = I_0(z)$ we obtain

$$\begin{aligned} F^{(3)}\left[0, \frac{u^2}{4}, \frac{v^2}{4}\right] &= F^{0:0;0}_{1:0;0}\left[\begin{matrix}-: & -; & -; \\ 1: & -; & -;\end{matrix}\; \frac{u^2}{4},\ \frac{v^2}{4}\right] \\ &= {}_0F_1\left[-;\ 1;\ \frac{u^2}{4}+\frac{v^2}{4}\right] = I_0\left(\sqrt{u^2+v^2}\right); \end{aligned}$$

and equations (21) and (22) reduce to the Carlton-Kimball distribution in Cartesian coordinates

$$p_t(x,y) = \frac{1}{2\pi\sigma^2} \exp\left(\frac{-t^2\nu_0^2}{2\sigma^2}\right) \exp\left(-\frac{x^2+y^2}{2\sigma^2}\right) I_0\left(\frac{t\nu_0}{\sigma^2}\sqrt{x^2+y^2}\right).$$

Now switching to polar coordinates we see that this result is just equation (13) since $x^2+y^2=r^2$ and the Jacobian of the transformation is r.

Thus equations (21) and (22) give the target's future position density $p_t(x,y)$ assuming an initial bivariate normal position distribution, constant speed and uniform direction of motion. We remark that the analysis employed above also shows that when $\sigma_x = \sigma_y = \sigma$, then equations (19) and (20) reduce to equation (10).

The particular case of Srivastava's $F^{(3)}$-function which appears in the previous results of this section may be expressed in terms of Kampé de Fériet functions in three different ways. These expressions are recorded as follows:

$$\begin{aligned} F^{(3)}[x,\ y,\ z] &= \sum_{n=0}^{\infty} \frac{(1/2)_n}{(1)_n} \frac{x^n}{n!} F^{0:1;0}_{1:1;0}\left[\begin{matrix} \text{——}: & 1/2+n; & \text{—}; \\ 1+n: & 1/2; & \text{—}; \end{matrix}\ z,\ y\right] \\ &= \sum_{n=0}^{\infty} \frac{1}{(1)_n} \frac{y^n}{n!} F^{1:0;0}_{1:1;0}\left[\begin{matrix} 1/2: & \text{—}; & \text{—}; \\ 1+n: & 1/2; & \text{—}; \end{matrix}\ z,\ x\right] \\ &= \sum_{n=0}^{\infty} \frac{1}{(1)_n} \frac{z^n}{n!} F^{0:1;0}_{1:0;0}\left[\begin{matrix} \text{——}: & 1/2+n; & \text{—}; \\ 1+n: & \text{——}; & \text{—}; \end{matrix}\ x,\ y\right]. \end{aligned} \tag{23}$$

Equation (23) may be rewritten also in terms of the function Φ_3 which is one of the seven confluent forms of the Appell functions [8, p. 58]. Thus

$$F^{(3)}[x,\ y,\ z] = \sum_{n=0}^{\infty} \frac{1}{(1)_n} \frac{z^n}{n!} \Phi_3[1/2+n;\ 1+n;\ x,\ y].$$

7. CONCLUSIONS

In this paper, we have examined the problem of determining the position probability density of a target's future location, assuming circular and bivariate Gaussian distributed estimates of its initial position, uniformly likely speed and direction in some interval, and a constant velocity. Exact results for this density have been given in closed form in various ways in terms of Kampé de Fériet functions and other generalized hypergeometric functions. These functions are efficiently computable and should prove useful in determining and developing optimal search plans in many search applications.

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